Diferenciabilidade em Superfícies

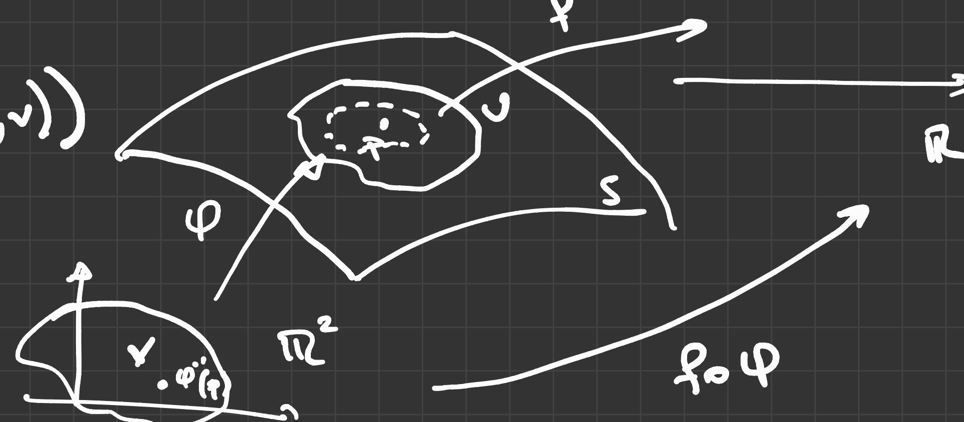
$f: \mathbb{R}^n \longrightarrow S \subset \mathbb{R}^3$ ⇝ sabemos o significado de diferenc. de FVV

$f: S \supset U \longrightarrow \mathbb{R}$

$p \in U$

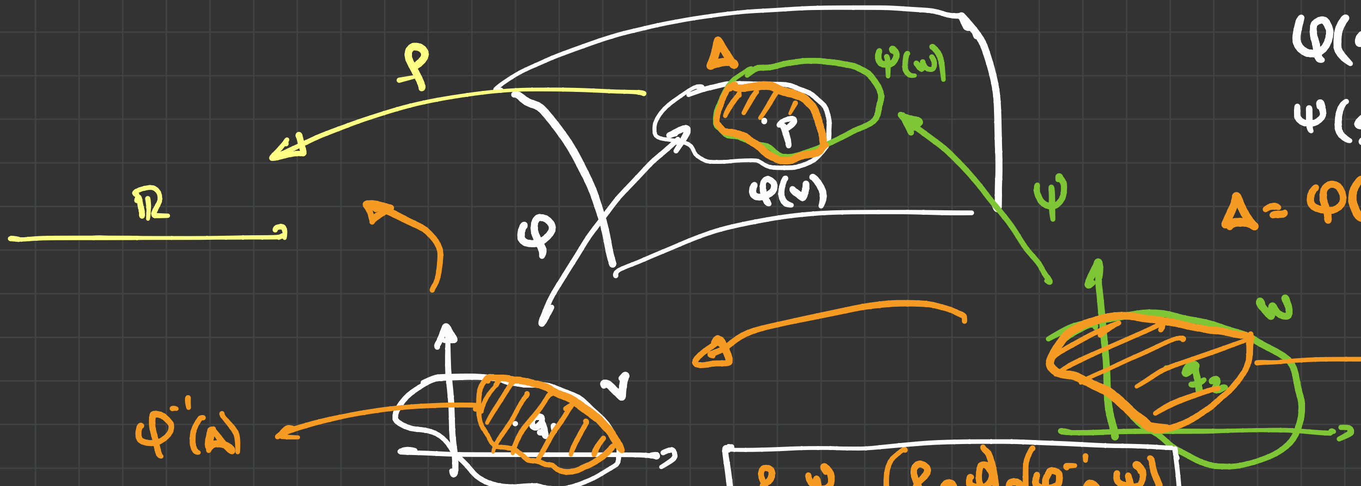
$\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$

$\underline{f \circ \varphi}: \underset{\mathbb{R}^2}{\varphi^{-1}(U)} \longrightarrow \mathbb{R}$



Def: $\boxed{f : U \subset S \to \boxed{\mathbb{R}^n}}$, U aberto de S, S sup. reg.

f é diferenciável em p se $f \circ \varphi$ é dif. em $\varphi^{-1}(p)$ onde $\varphi : V \to S$ é uma parametrização com $p \in \varphi(V)$ de S.



$\varphi(q_1) = p$

$\psi(q_2) = p$

$A = \varphi(V) \cap \psi(W)$

$\boxed{f \circ \psi = (f \circ \varphi) \circ (\varphi^{-1} \circ \psi)}$

f dif $p \iff f \circ \varphi$ dif. em $q_1 \overset{(*)}{\iff} f \circ \psi$ é dif em q_2

Prop (Mudança de Parâmetros):

$p \in S$, S sup. reg. e $\varphi: V \subset \mathbb{R}^2 \to S$ e $\psi: W \subset \mathbb{R}^2 \to S$ duas params. tq $p \in \varphi(V) \cap \psi(W) = \Delta$.

Então $h = \varphi^{-1} \circ \psi : \psi^{-1}(\Delta) \longrightarrow \varphi^{-1}(\Delta)$ é difeomorf. ou seja, é dif. com inversa dif. (C^∞).

[h: mudança de parâmetros]

Dem: $h = \underbrace{\boxed{\varphi^{-1}}}_{F^{-1}} \circ \underbrace{\psi}_{C^\infty}$ é homeomorfismo (hip. 2 sup.)

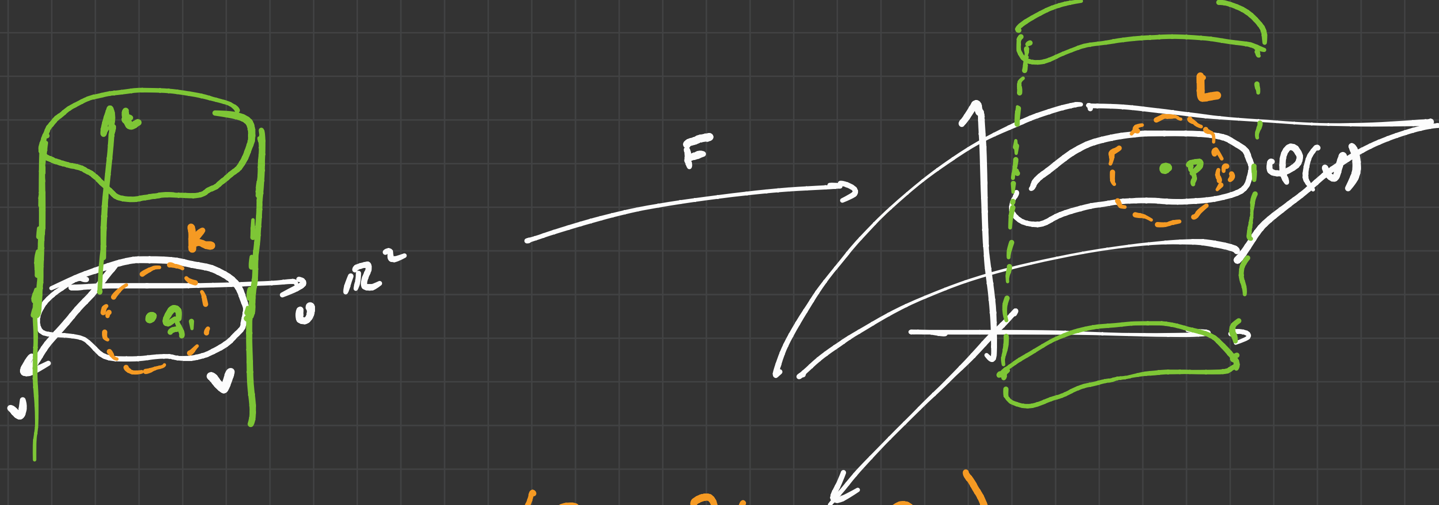
Suponha SPG

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

$$\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$$

$F: V \times \mathbb{R} \longrightarrow \mathbb{R}^3$ ("$\mathbb{R}^3 \to \mathbb{R}^3$")

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v) + t)$$



$$dF_{q_1} = \left.\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{pmatrix}\right|_{q_1} =: M$$

$$\det M = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

Pelo TFI F é difeomorfismo numa viz de q_1

$F|_K$ é difeo

$F^{-1}: L \to K$ é dif.

$h^{-1} = \psi^{-1} \circ \varphi$

Mas $h = \varphi^{-1} \circ \psi = \underbrace{F^{-1}}_{\substack{C^\infty \\ (\text{TFI})}} \circ \underbrace{\psi}_{\substack{C^\infty \\ (\text{hip. sup.})}}$

Logo h é C^∞

Numa viz de q, h é difeomorfismo. ⊠

Obs 1: Se $\varphi : U \subset \mathbb{R}^2 \to S$ é parametrização diferenciável.

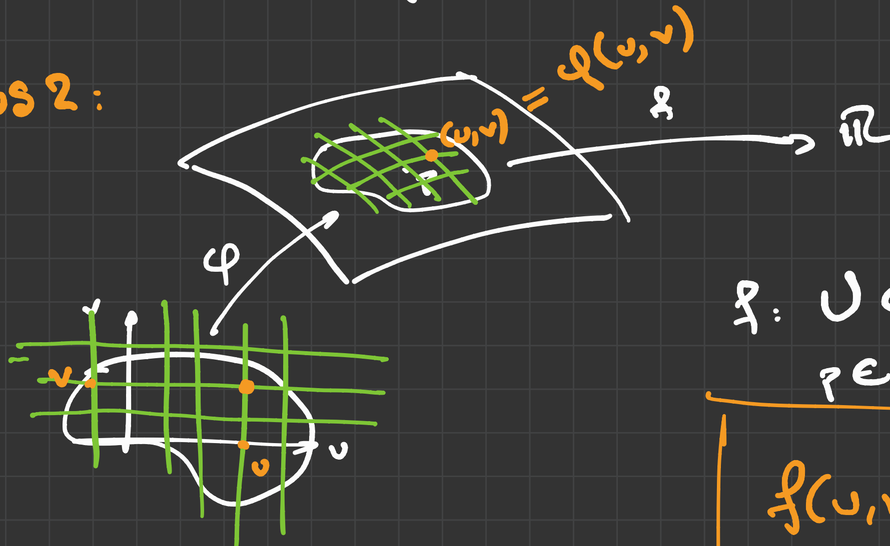
então $\varphi^{-1} : \varphi(U) \subset S \to \mathbb{R}^{\boxed{2}}$ é $n=2$

$\boxed{\varphi^{-1} \text{ é dif}} \iff \varphi^{-1} \circ \psi$ é dif. p/ alguma param. ψ de S

$\psi = \varphi$ $\quad \varphi^{-1} \circ \varphi = \text{Id} : U \to U$ dif

Obs 1': Se $\varphi: U \to S$ é param. então φ é difeomorfismo. (hip 1. sup + obs 1).

obs 2:

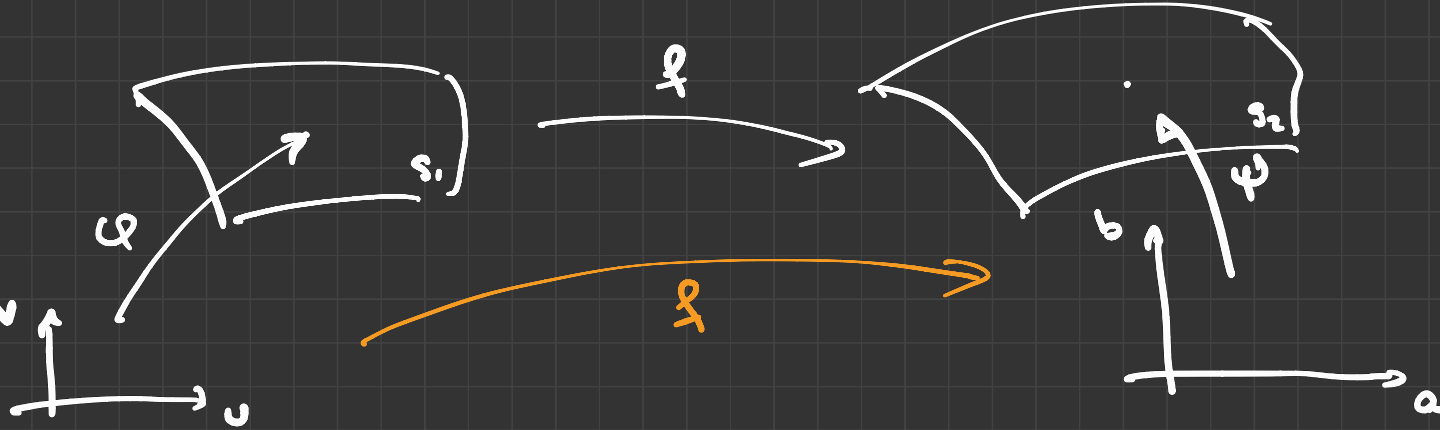


$f: U \subset S \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$

$p \in U$

$$\boxed{f(u,v) = f \circ \varphi(u,v)}$$

Obs 2' :

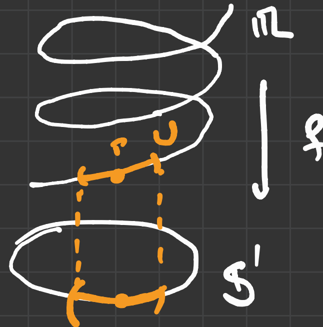


$$\tilde{f} = \psi^{-1} \circ f \circ \varphi$$

$$= \left(f_1(u,v) \, , \, f_2(u,v) \right)$$

Obs 3: Def: $f: X \longrightarrow Y$ é difeomorfismo local se $\forall x \in X$, $\exists U$ viz de x t.q. $f|_U$ é difeomorfismo

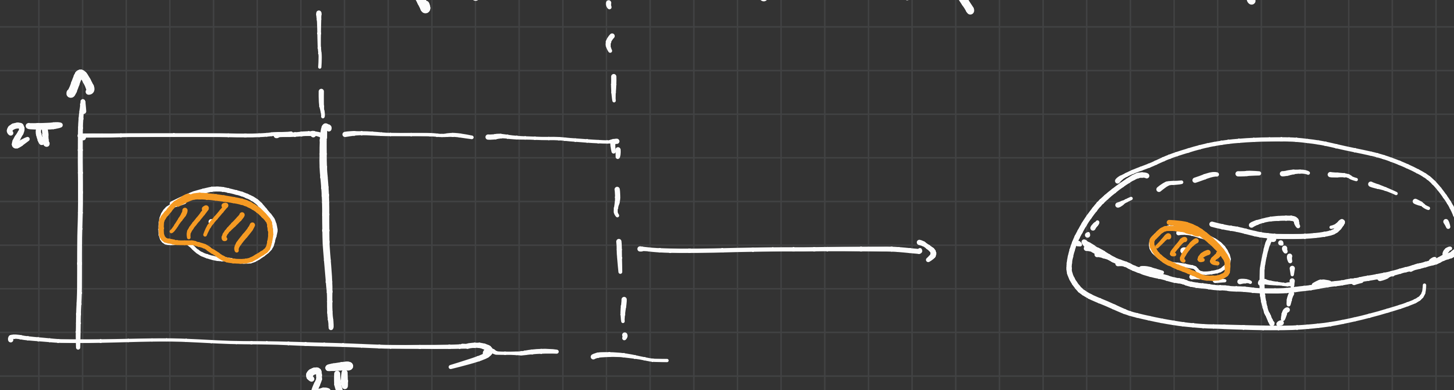
Ex 1:



$$f : \mathbb{R} \longrightarrow S^1$$
$$t \longmapsto (\cos 2\pi t, \operatorname{sen} 2\pi t)$$

Ex 2:

$$f(u,v) = \big((a\cos u + r)\cos v,\ (a\cos u + r)\operatorname{sen} v,\ a \operatorname{sen} u \big)$$



f é difeomorf. local mas não difeo. global

Ex 1: S sup. reg.

$V \subset \mathbb{R}^3$ aberto tq $S \subset V$.

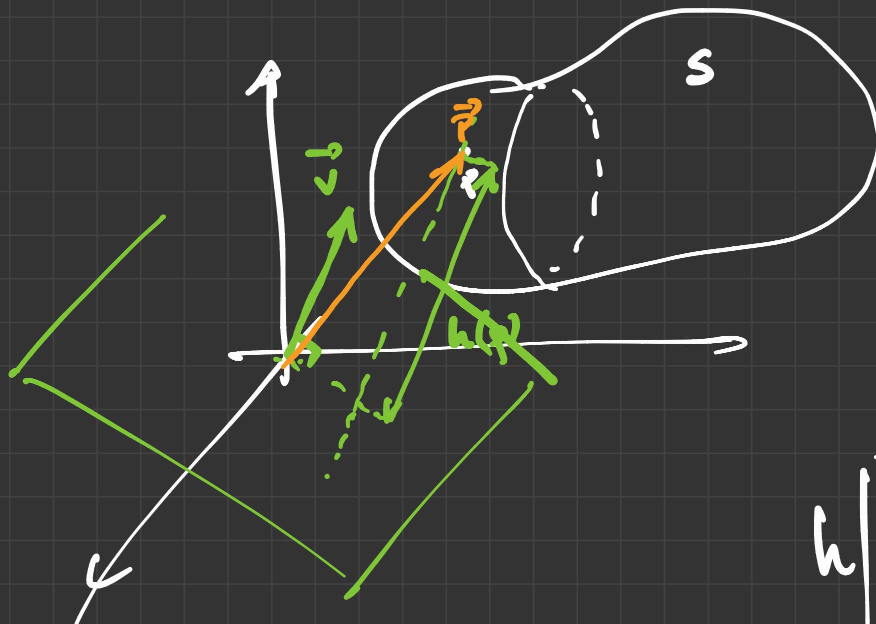
Se $\boxed{f : V \longrightarrow \mathbb{R}}$ é diferenciável então f restrita a S é função diferenciável sobre S.

Dem: Seja $p \in S$ e φ param. numa viz de p

$$\boxed{f\big|_S \text{ é dif em } p} \Longleftrightarrow f\big|_S \circ \varphi \text{ dif em } q \text{ (onde } \varphi(q) = p)$$

Mas $f\big|_S \circ \varphi = \underbrace{f}_{C^\infty} \circ \underbrace{\varphi}_{C^\infty}$ é C^∞

i) Função altura



$\|\vec{v}\| = 1$

$h(p) = \vec{v} \cdot \vec{p}$

$$\boxed{\begin{aligned} h : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ p &\longmapsto v \cdot p \end{aligned}}$$

$h\big|_S$ é dif

ii) $p_0 \in \mathbb{R}^3$

$$d : \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad C^1$$
$$p \longmapsto |p - p_0|^2$$

$$d\Big|_S \text{ é } C^1$$

Ex 1': $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^n$ dif $\Longrightarrow f\Big|_S$ dif

Ex 1'': $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tq $f(S_1) \subset S_2$

f dif $\Longrightarrow f : S_1 \longrightarrow S_2$ é dif.

iii) $\begin{array}{ccc} \mathbb{R}^3 & & \mathbb{R}^3 \\ (x,y,z) & \xmapsto{f} & (-x,-y,-z) \end{array}$

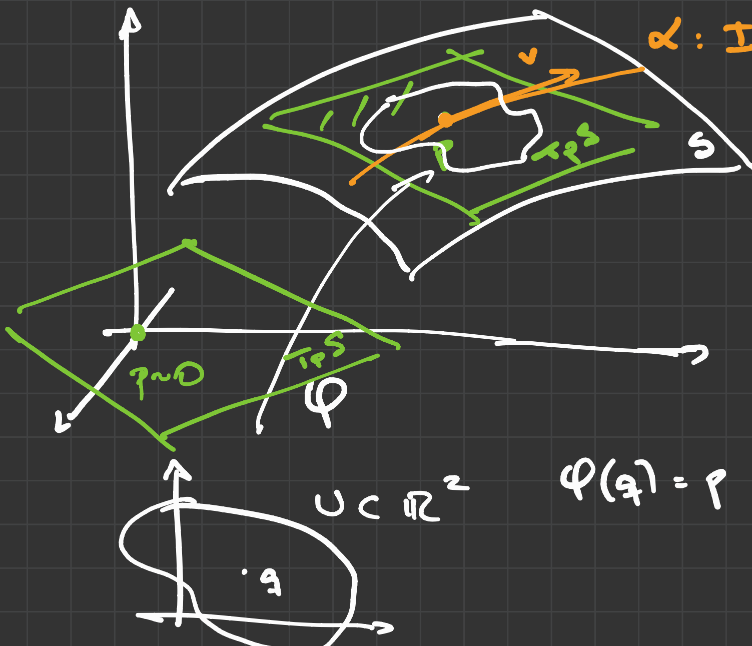
$$f\big|_{S^2} : S^2 \longrightarrow S^2$$

iv) $(x,y,z) \xmapsto{g} (ax, by, cz)$

$x^2+y^2+z^2 = 1$ $\qquad \dfrac{(ax)^2}{a^2} + \dfrac{(by)^2}{b^2} + \dfrac{(cz)^2}{c^2} = 1$

$$g\big|_{S^2} : S^2 \longrightarrow E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

Plano Tangente



$\alpha : I \subset \mathbb{R} \longrightarrow S$

$\alpha(0) = p$

$v = \alpha'(0) \in T_pS$

$f : V \subset S \to \mathbb{R}$

$$\nabla_v f = \frac{d f(\alpha(t))}{dt}$$

$v \sim \nabla_v$

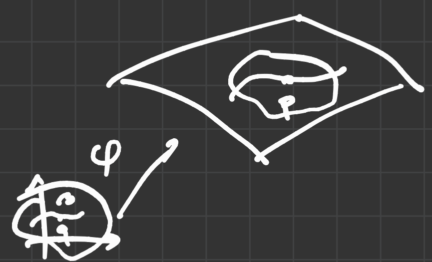
$$T_p S = d\varphi_q(\mathbb{R}^2)$$

Prop 1: $\varphi : U \subset \mathbb{R}^2 \longrightarrow S$ param. de S, sup. reg.,
$q \in U$ tq $\varphi(q) = p$. O subesp. vetorial de dim. 2,

$$T_pS = d\varphi_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincide com o cjto dos vetores tangentes
a S em p.

Dem.: Seja w vet. tg. a S em p, ou seja,
tq $\exists\ \alpha : I \subset \mathbb{R} \longrightarrow S$ com $\alpha(0) = p$ e $\alpha'(0) = w$
curva reg.



$$\varphi^{-1} \circ \alpha(t) = \beta(t)$$

Pela definição de diferencial:

$$d\varphi_q(\beta'(0)) = w$$

$$d\varphi_q(\beta'(0)) = \frac{d}{dt}\varphi\circ\beta\Big|_{t=0} = \alpha'(0) = w$$

Seja $w = d\varphi_q(v)$, $v \in \mathbb{R}^2$

$$\underline{\beta(t) = q + tv} \qquad \beta'(0) = v \qquad \beta(0) = q$$

$$w = d\varphi_q(\beta'(0)) = \frac{d}{dt}\underbrace{\varphi\circ\beta}_{\alpha}\Big|_{t=0} = \alpha'(0)$$

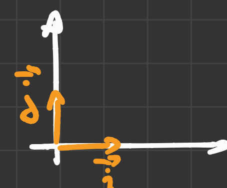
$\boxed{/\!/}$

Obs 1: Prop 1 $\Longrightarrow$ T_pS não depende da param. φ escolhida.

Obs 2: $\varphi(u,v) = \left(x(u,v),\, y(u,v),\, z(u,v)\right)$

$$\frac{\partial \varphi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

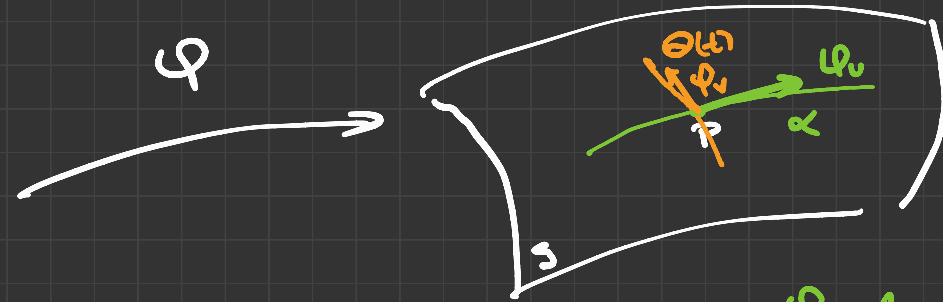
$\varphi(q) = p$



$$\frac{\partial x}{\partial u}(q) = \frac{d}{dt}\, x(\beta(t))\Big|_{t=0}$$

onde

$\beta(t) = q + t\vec{i}$



$$\gamma(t) = q + t\vec{\partial}$$

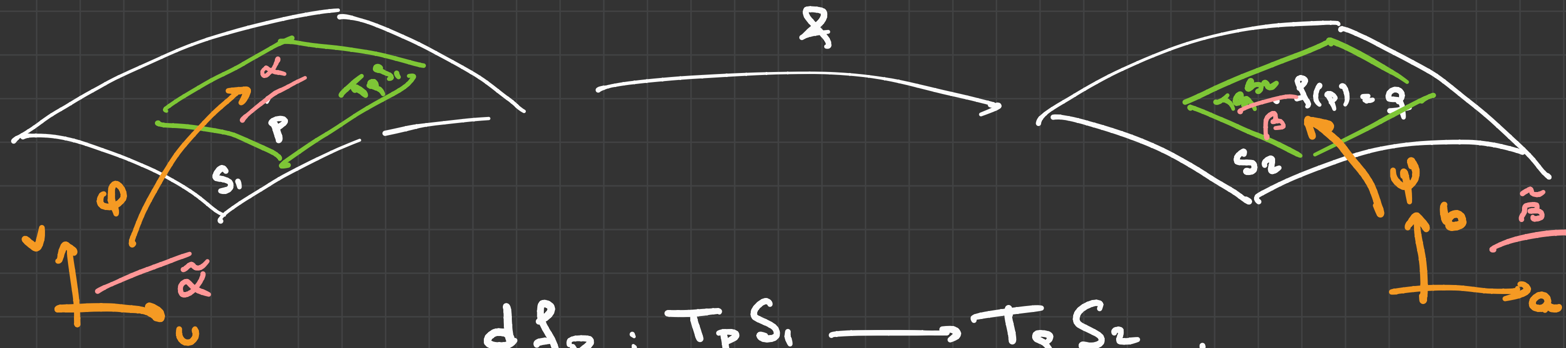
$$\alpha = \varphi \circ \beta$$

$$\alpha'(0) = \frac{\partial \varphi}{\partial u}(q) = \varphi_u(q)$$

$$\theta = \varphi \circ \gamma$$

$$\theta'(0) = \frac{\partial \varphi}{\partial v}(q) = \varphi_v(q)$$

$$d\varphi_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix}$$



$$df_p : T_pS_1 \longrightarrow T_qS_2$$

$$\alpha'(0) = w \longmapsto (f\circ\alpha)'(0) = df_p(w)$$

$$\alpha : I \subset \mathbb{R} \to S_1 \qquad f\circ\alpha = \beta \qquad \beta : I \to S_2$$

$$\beta(0) = q$$

Prop 2: Dado w como acima, $\beta'(0)$ não depende da escolha de α. Além disso, df_p é linear.

Dem: Sejam $\varphi(u,v)$ e $\psi(a,b)$ params. de S_1 e S_2 em vizs. de p e q respectivamente.

$$\underbrace{\psi^{-1}\circ f\circ\varphi}_{\mathbb{R}^2\to\mathbb{R}}(u,v) \to f(u,v) = \left(f_1(u,v),\ f_2(u,v)\right)$$

Seja $\underbrace{\alpha(t)}_{\tilde{\alpha}\,=\,\varphi^{-1}\circ\alpha} = (u(t), v(t))$

$$\tilde{\alpha}'(0) = (u'(0), v'(0))_{\vec{i},\vec{j}}$$

$$\alpha'(0) = u'(0)\varphi_u + v'(0)\varphi_v = (u'(0), v'(0))_{\varphi_u,\varphi_v}$$

$\beta = f\circ\alpha$: $\underbrace{\beta(t)}_{\tilde{\beta}\,=\,\psi^{-1}\circ\beta} = \left(f_1(u(t),v(t)),\ f_2(u(t),v(t))\right)$

$$\tilde{\beta}'_{i,j}(0) = \beta'_{\psi_a,\psi_b}(0) = \left(\frac{\partial f_1}{\partial u}u'(0) + \frac{\partial f_1}{\partial v}v'(0),\ \frac{\partial f_2}{\partial u}u'(0) + \frac{\partial f_2}{\partial v}v'(0)\right)$$

$$w = df_p(v)$$

$$\| $$

$$\beta'(0) = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}}_{df_p} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

$\beta'(0) \to B_2 = \{\psi_a, \psi_b\}$

$\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} \to \alpha'(0)$ $\quad B_1 = \{\varphi_u, \varphi_v\}$

$$\tilde{\beta}(0) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

$\tilde{\beta}(0) \to B = \{i, j\}$

$d(\psi^{-1} \circ f \circ \varphi)_{\varphi^{-1}(p)}$ $\quad \tilde{\alpha}(0)_{B = \{i, j\}}$

$\downarrow$

Base can.
de $\mathbb{R}^2$