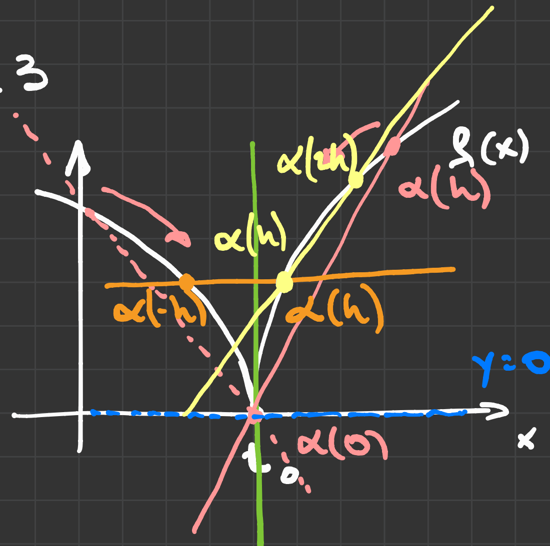
Seção 1.3

7)



$t_0 = 0$

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & \text{se } x \geq 0 \\ \sqrt[3]{(-x)^2}, & \text{se } x < 0 \end{cases}$$

$$s = \sqrt[3]{s^2}$$

$$\alpha(t) = (t, f(t)) = (t^3, t^2)$$

Existe tangente fraca

Não há tang. forte em t_0

$k = 2h$

$k = -h$

$h \to 0$

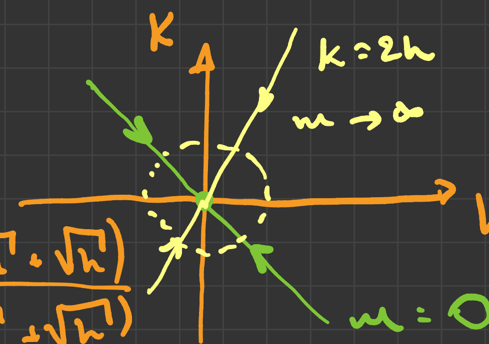
$$\frac{\alpha(h) - \alpha(-h)}{2h} = 0$$

$$= m$$

$$\frac{\alpha(2h) - \alpha(h)}{h}$$

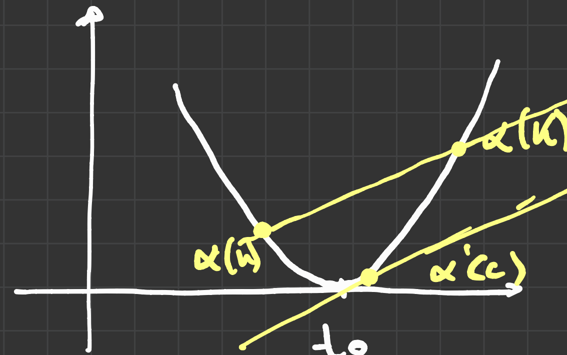
$$= \frac{\sqrt{2h} - \sqrt{h}}{h} \cdot \frac{(\sqrt{2h} + \sqrt{h})}{(\sqrt{2h} + \sqrt{h})}$$

$$\ldots = \frac{h}{h(\sqrt{2h} + \sqrt{h})} \xrightarrow{h \to 0} \infty$$



$$\boxed{\left|\frac{\alpha(h)-\alpha(K)}{h-K}\right|}$$

$$\frac{\alpha(0)-\alpha(K)}{0-K}$$



$\underline{\text{TVM}}$

$\exists\, c \in (h,k)$ tq $\dfrac{\alpha(k)-\alpha(h)}{k-h} = \alpha'(c)$

$h,k \to 0$

$c \to 0 \qquad \alpha'(c) \to 0$

α curva C^n, $n \geq 1 \Rightarrow$ vale TVM $\Rightarrow \exists$ tangente fake

"

tangente fisica

$$f(h,k) = \frac{\alpha(h) - \alpha(k)}{h-k}$$

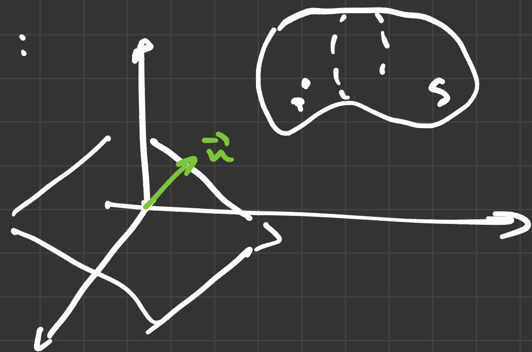
$$\tilde{\alpha}(t) = (t, \alpha(t))$$

$\shortparallel$

$m_{h,k}$

(Continuando: Diferencial de função entre Superfícies)

Ex1:



$$f: S \longrightarrow \mathbb{R}$$
$$p \longmapsto \vec{n} \cdot p$$

$$w \in T_pS \xrightarrow{\;df_p\;} \,?$$

$$df_p : T_pS \to \mathbb{R}$$

$$\alpha : I \to S$$

$$\boxed{\begin{array}{l} \alpha(0) = p \\ \alpha'(0) = w \end{array}}$$

$$(f \circ \alpha)(t) = \vec{n} \cdot \alpha(t)$$
$$= n_1 \alpha_1(t) + n_2 \alpha_2(t) + n_3 \alpha_3(t)$$

$$f \circ \alpha : I \longrightarrow \mathbb{R}$$

$$df_p(w) = \frac{d(f\circ\alpha)}{dt}(0)$$

$$(f\circ\alpha)'(0) = \vec{n} \cdot \alpha'(0)$$

Logo $df_p(w) = \vec{n} \cdot \alpha'(0) = \vec{n} \cdot w$

$$f : S \longrightarrow \underset{\substack{\parallel \\ M^n}}{\mathbb{R}^n}$$

M^n

$$f : \begin{cases} S \longrightarrow \mathbb{R}^n \\ T_pS \longrightarrow T_q\mathbb{R}^n = \mathbb{R}^n \\ S_1 \longrightarrow S_2 \\ T_pS_1 \to T_pS_2 \\ \mathbb{R}^m \longrightarrow S \\ T_p\mathbb{R}^m = \mathbb{R}^m \to T_qS \\ \mathbb{R}^m \longrightarrow \mathbb{R}^n \end{cases}$$

$$\underset{\substack{\parallel \\ \mathbb{R}^m}}{T_p\mathbb{R}^m} \longrightarrow \underset{\substack{\parallel \\ \mathbb{R}^n}}{T_q\mathbb{R}^n}$$

df_p

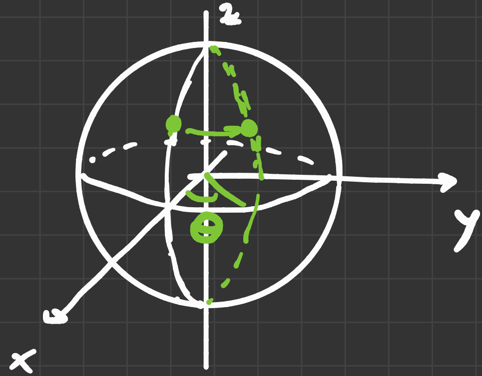
$f : S \longrightarrow \mathbb{R}$

$df_p : T_pS \longrightarrow T_q\mathbb{R} = \mathbb{R}$

onde $q = f(p)$

$T_q\mathbb{R}^2 = \mathbb{R}^2$

Ex :



$\nearrow R_{z,\theta} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$R_{z,\theta} : S^2 \longrightarrow S^2$

$w \in T_pS \xrightarrow{(dR_{z,\theta})_p} \, ?$

T_qS

$\alpha(0) = p \quad \alpha'(0) = w$

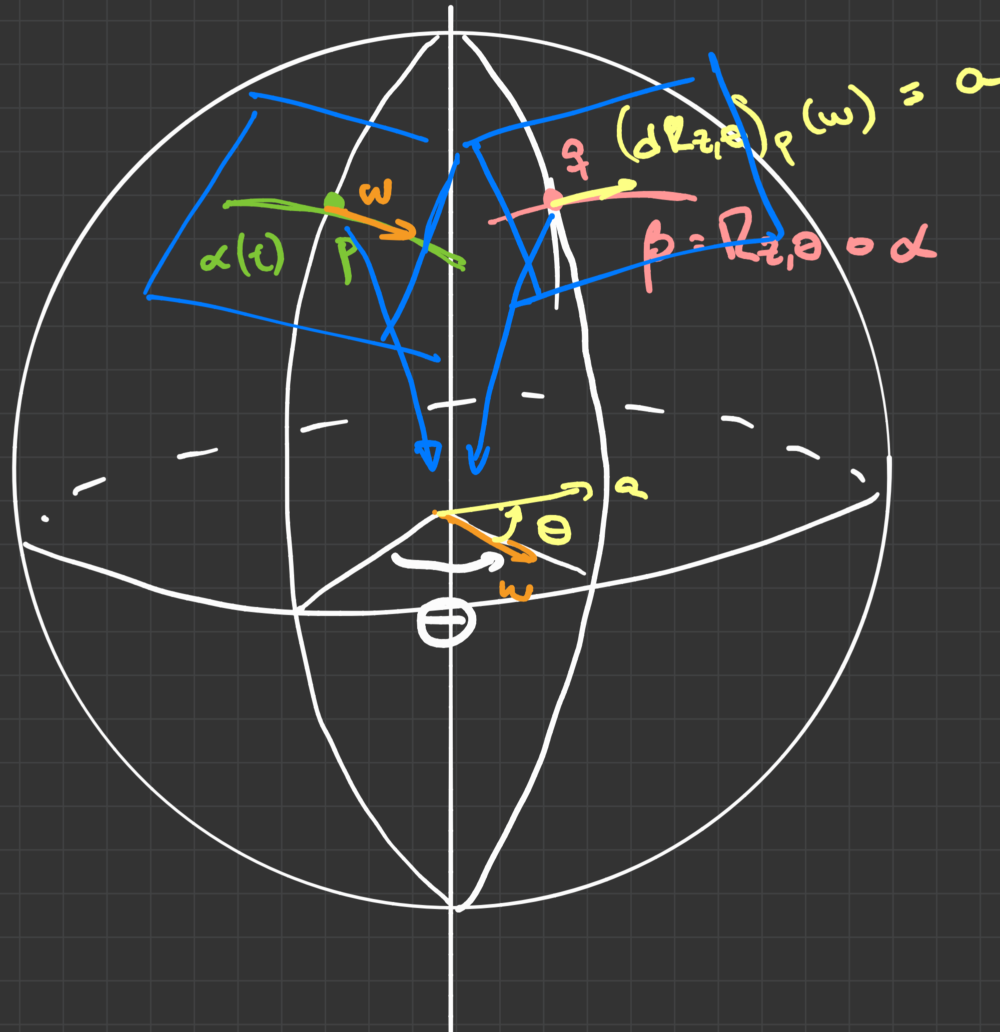
$q = R_{z,\theta}(p)$

$$(dR_{z,\theta})_p(w) = (\underline{R_{z,\theta} \circ \alpha})'(0)$$

$$= (dR_{z,\theta})_{\alpha(0)} \cdot \alpha'(0)$$

$$= R_{z,\theta} \cdot \alpha'(0) = R_{z,\theta} \cdot w$$

$$\underset{Q}{\shortparallel}$$



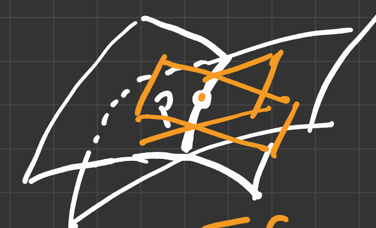
$(dR_{z,\theta})_p(\omega) = a$
q
ω
$\alpha(t)$
p
$\beta = R_{z,\theta} \circ \alpha$
a
θ
ω
θ

Obs 1:

T.F. Inv (Sup): Se S_1 e S_2 são sups. regs. e $f: S_1 \to S_2$ tem df_p isomorfismo entre T_pS_1 e $T_{f(p)}S_2$ então $\exists\, U \subset S_1$ aberto com $p \in U$ tq $f\big|_U$ é um difeomorfismo

Obs 2: Se φ é carta de S então

φ_u, φ_v é base de T_pS



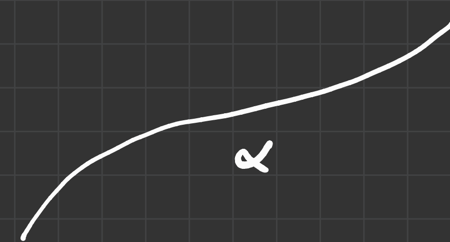
T_pS_1

T_pS_2

$$N(p) = \frac{\varphi_u \wedge \varphi_v}{|\varphi_u \wedge \varphi_v|} \to \text{vetor normal unitário a } S \text{ em } p$$

Primeira Forma Fundamental

$$\alpha : \underset{\underset{[a,b]}{\shortparallel}}{I} \to \mathbb{R}^3 \quad \text{curva regular}$$



$$C = \int_a^b |\alpha'(t)|\, dt$$

$$= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle}\, dt$$

$$|v|^2 = \langle v, v \rangle = v \cdot v$$

↑ prod. escalar

$$d(p,q) = \inf_{\alpha \text{ é curva q liga p a q}} C_\alpha$$

$S \subset \mathbb{R}^3$ sup. reg.

T_pS é subesp. vet. de $\mathbb{R}^3$

$v, w \in T_pS \subset \mathbb{R}^3$: $\langle v, w \rangle_{T_pS} = v \cdot_{\mathbb{R}^3} w$

Forma Quadrática:

$I_p(w) = \langle w, w \rangle = |w|^2 \geq 0$

$w \in T_pS$

$I_p : T_pS \to \mathbb{R}$

$\varphi : U \subset \mathbb{R}^2 \longrightarrow S$ sist. de coord.

$p \in \varphi(U)$

Prod. Int.

- $\langle v, w \rangle = \langle w, v \rangle$
- $\langle \alpha a + \beta b, v \rangle = \alpha \langle a, v \rangle + \beta \langle b, v \rangle$
- $\langle v, v \rangle \geq 0 \ \forall v$

  $= 0 \iff v = 0$

$w \in T_pS: \quad w = w_1 \varphi_u + w_2 \varphi_v$

$$I_p(w) = \langle w, w \rangle = \langle w_1 \varphi_u + w_2 \varphi_v, w_1 \varphi_u + w_2 \varphi_v \rangle$$

$$= w_1^2 \langle \varphi_u, \varphi_u \rangle + 2 w_1 w_2 \langle \varphi_u, \varphi_v \rangle + w_2^2 \langle \varphi_v, \varphi_v \rangle$$

$$= w_1^2 E + 2 w_1 w_2 F + w_2^2 G$$

$$\boxed{\begin{aligned} E_\varphi &= \langle \varphi_u, \varphi_u \rangle \\ F_\varphi &= \langle \varphi_u, \varphi_v \rangle \\ G_\varphi &= \langle \varphi_v, \varphi_v \rangle \end{aligned}}$$

Def: A forma quadrática $I_p : T_pS \to \mathbb{R}$ definida por $I_p(w) = \langle w, w \rangle$ é chamada Primeira Forma Fundamental de S em p.

Obs: $|v| = \sqrt{\langle v, v\rangle}$

$$\underset{\displaystyle |v+w|^2}{\underset{\shortparallel}{\langle v+w, v+w\rangle}} = \underset{\displaystyle |v|^2}{\underset{\shortparallel}{\langle v,v\rangle}} + 2\langle v,w\rangle + \underset{\displaystyle |w|^2}{\underset{\shortparallel}{\langle w,w\rangle}}$$

$$\langle v,w\rangle = \frac{|v+w|^2 - |v|^2 - |w|^2}{2}$$

"I_p carrega a informação do prod. interno de T_pS dentro dela"

Obs 2: $v, w \in T_pS$

$$\langle v,w\rangle = |v||w|\cos\theta$$

$$\cos\theta = \frac{\langle v,w\rangle}{|v|\,|w|} \qquad \theta \in [0,\pi]$$

$$\theta = \pi/2 \iff \langle v,w\rangle = 0$$

Se $F = \langle \varphi_u, \varphi_v\rangle = 0$ $\quad$ φ é parametriz. ortogonal

Ex 1: (Plano em $\mathbb{R}^3$)

$$\Pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto P + u\vec{w_1} + v\vec{w_2}$$

$P \in \Pi$, w_1, w_2 são vets. diretores de Π

Π_u, Π_v?

$$\Pi_u = \frac{\partial \Pi}{\partial u} = \vec{w_1}$$

$$\Pi_v = \frac{\partial \Pi}{\partial v} = \vec{w_2}$$

Se w_1 e w_2 são base orton.:

$$E = G = 1 \quad F = 0$$

Ex 2:

$$\varphi : (0, 2\pi) \times \mathbb{R} \longrightarrow S$$
$$(u, v) \longmapsto (\cos u, \operatorname{sen} u, v)$$

$$\varphi_u = \frac{\partial \varphi}{\partial u} = (-\operatorname{sen} u, \cos u, 0)$$

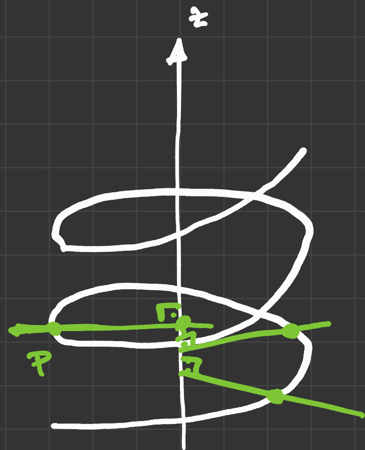
$$\varphi_v = \frac{\partial \varphi}{\partial v} = (0, 0, 1)$$

$$E = |\varphi_u|^2 = \operatorname{sen}^2 u + \cos^2 u = 1$$

$$F = \langle \varphi_u, \varphi_v \rangle = 0$$

$$G = |\varphi_v|^2 = 1$$

Ex 3: (Helicóide)



$(\cos u, \operatorname{sen} u, u) \rightsquigarrow$ hélice

$p \in$ Hélice toma r_p reta $\perp$ ao eixo z por p

$r_p : (0,0,u) + v(\cos u, \operatorname{sen} u, 0)$

$$\varphi : \mathbb{R}^2 \longrightarrow S$$
$$(u,v) \longmapsto \big(v\cos u, v \operatorname{sen} u, u\big)$$

$$\varphi_u = \frac{\partial \varphi}{\partial u} = \left(-v \operatorname{sen} u, v\cos u, 1\right)$$

$$\varphi_v = \frac{\partial \varphi}{\partial v} = \left(\cos u, \operatorname{sen} u, 0\right)$$

$$E = v^2(\operatorname{sen}^2 u + \cos^2 u) + 1 = v^2 + 1$$

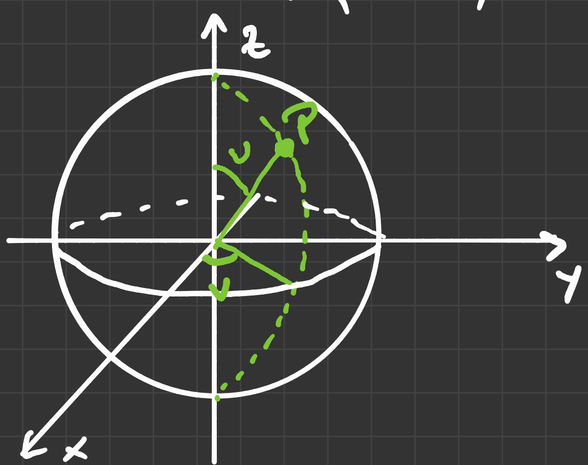
$$F = 0$$

$$G = 1$$

Ex 4: (Esfera)

$$(u,v) \in (0,\pi) \times (0,2\pi)$$

$$\varphi(u,v) = (\operatorname{sen} u \cos v, \operatorname{sen} u \operatorname{sen} v, \cos u)$$



$$\varphi_u = (\cos u \cos v, \cos u \operatorname{sen} v, -\operatorname{sen} u)$$

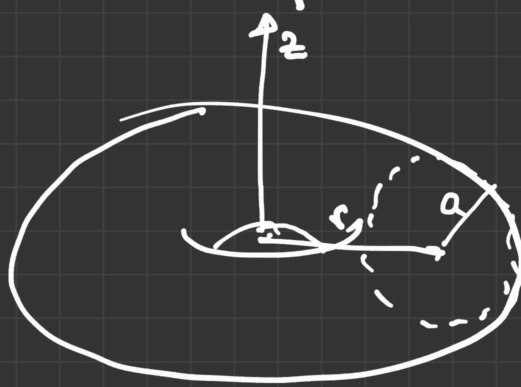
$$\varphi_v = (-\operatorname{sen} u \operatorname{sen} v, +\operatorname{sen} u \cos v, 0)$$

$$E = \cos^2 u \, (\underbrace{\cos^2 v + \operatorname{sen}^2 v}_{1}) + \operatorname{sen}^2 u = 1$$

$$F = 0$$

$$G = \operatorname{sen}^2 u \left(\operatorname{sen}^2 v + \cos^2 v \right) = \operatorname{sen}^2 u$$

Ex 5: $\varphi(u,v) = \Big((r + a\cos u)\cos v, \; (r + a\cos u)\operatorname{sen} v, \; a \operatorname{sen} u \Big)$



$\varphi_u = \left(-a \operatorname{sen} u \cos v, \; -a \operatorname{sen} u \operatorname{sen} v, \; a\cos u \right)$

$\varphi_v = \Big(-(r + a\cos u)\operatorname{sen} v, \; (r + a\cos u)\cos v, \; 0 \Big)$

$$E = (a^2 \operatorname{sen}^2 u)\,(\underbrace{\operatorname{sen}^2 v + \cos^2 v}_{1}) + a^2 \cos^2 u = a^2$$

$$F = 0$$

$$G = (r + a\cos u)^2$$

Obs 1: (Comprimento de curvas em coords.)

φ sist. de coord.

$$\alpha(t) = \varphi \circ \tilde{\alpha}(t) \qquad \text{onde } \tilde{\alpha}(t) = \underset{\mathbb{R}^2}{\underset{\cap}{(u(t), v(t))}}$$

$$\alpha'(t) = \varphi_u \cdot u'(t) + \varphi_v \cdot v'(t)$$

$$I_p(\alpha'(t)) = |\alpha'(t)|^2 = \langle u'\varphi_u + v'\varphi_v, u'\varphi_u + v'\varphi_v \rangle = \dots$$

$$\dots = u'^2 E + 2u'v'F + v'^2 G$$

$$C = \int_a^b \sqrt{I_p(\alpha'(t))}\, dt = \int_a^b \sqrt{u'^2 E + 2u'v'F + v'^2 G}\, dt$$

$$\left(\frac{du}{dt}\right)^2 E + 2\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) F + \left(\frac{dv}{dt}\right)^2 G = \left(\frac{ds}{dt}\right)^2$$

onde $s(t)$ é o comprimento de arco:

$$s(t) = \int_a^t |\alpha'(\lambda)|\, d\lambda$$

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

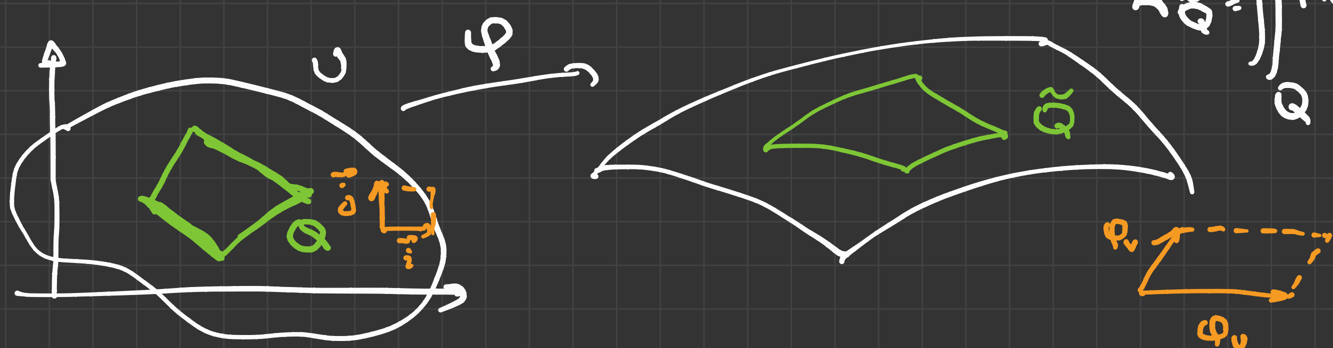
Obs 2:

Def: Um **domínio** (regular) de S é um aberto conexo de S cuja fronteira é homeomorfa a um círculo S^1 (param. como "curva reg." cuja deriv. se anula num nº finito de pontos). Uma **região** de S é a união de um domínio com a sua fronteira.

Uma região é **limitada** se existe bola de $\mathbb{R}^3$ que a contenha.

$$A_{\tilde{Q}} = \iint_Q |\varphi_u \wedge \varphi_v| \, du\,dv$$

$$A_Q = \iint_Q du\,dv$$



$$|v \wedge w|^2 = |v|^2|w|^2 \sin^2\theta = |v|^2|w|^2(1-\cos^2\theta)$$

$$= |v|^2|w|^2 - \langle v, w\rangle^2$$

$$|\varphi_u \wedge \varphi_v|^2 = EG - F^2$$

$$A\bar{Q} = \iint_Q \sqrt{EG - F^2}\, du\, dv$$