Superfícies Regulares (Manfredo GD)

Def: $S \subset \mathbb{R}^3$ tq $\forall$ cada $p \in S$ existem vizs. $V \subset \mathbb{R}^3$ de p e $U \subset \mathbb{R}^2$ aberto tq

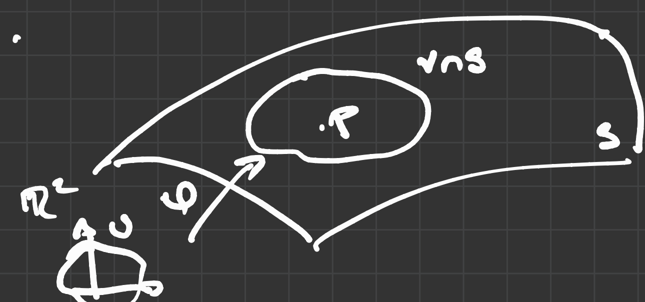
$\varphi : U \to V \cap S$

1) $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$

   φ é C^∞

2) φ é homeomorfismo. (φ^{-1} é invers. e cont.)

3) $d\varphi_q$ é injetora.



Prop 1: Se $f: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ é fç. dif.
então graf f é superfície regular

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & S \\ (x,y) & \longmapsto & (x,y,f(x,y)) \end{array}$$

Def: Seja $F: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad (\underset{\substack{\shortparallel \\ 1}}{m} < n)$

p é <u>ponto crítico</u> de F se
$dF_p: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ não é sobrejetiva

Se p é ponto crítico $F(p)$ é <u>valor crítico</u>

Se $q \in \mathbb{R}^m$ não é valor crítico de F então
q é <u>valor regular</u>.

$\hookrightarrow A = f^{-1}(\{q\}) = \{p \in U;\ F(p) = q\}$

q é valor regular $\Longleftrightarrow dF_p$ é sobre $\forall p \in A$

Prop 2: Se $f: U \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ dif e $a \in f(U)$ é valor regular então $f^{-1}(a)$ é sup. regular

Ex:

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad r > 0$$
$$(x,y,z) \longmapsto x^2+y^2+z^2$$

$$S_r^2 = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = r^2\} = f^{-1}(r^2)$$

$$df_{\underbrace{(x_0,y_0,z_0)}_{q}} = \left(\frac{\partial f}{\partial x}(q), \frac{\partial f}{\partial y}(q), \frac{\partial f}{\partial z}(q)\right)$$

$$= (\underbrace{2x_0, 2y_0, 2z_0}_{\vec{v}})$$

$$df_q(x) = \vec{v} \cdot x$$

$r > 0$ é valor regular

$p = (0,0,r)$

$p \in S^2(r)$

$$\boxed{df_p = (0,0,\boxed{2r})}$$

$$2r = \frac{\partial f}{\partial z}(p) \neq 0$$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad n = 1$

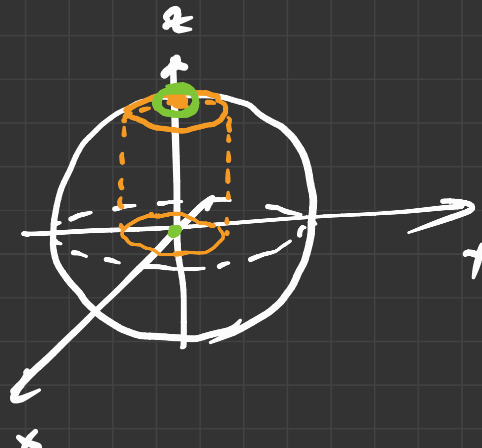
$(x,y,z) \longmapsto x^2+y^2+z^2$ $\quad m = 2$

$\longrightarrow$ Numa viz de p

S é gráfico de

$g : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto z$



$a = r$

$b = (0,0)$

$(b,a) = (0,0,r)$

$\quad = p$

$c = r^2$

TFImp $\Longrightarrow \exists\, G : b \in U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

t.q. $G(b) = a$ ~

$f(b, G(b)) = r^2$

$$G : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto G(x,y)$$

$$f(b, G(b)) = r^2$$

$$G(0,0) = r$$

$$f(x, y, G(x,y)) = r^2$$

$$\longrightarrow x^2 + y^2 + (G(x,y))^2 = r^2$$

$$G(x,y) = \sqrt{r^2 - x^2 - y^2}$$

Quádricas

Elipsoide: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

1 é valor regular

$$E = f^{-1}(1)$$

$$df_p = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$$

Paraboloide Elíptico: $z = x^2 + y^2$

$$f(x,y,z) = x^2 + y^2 - z$$

$$df_p = (2x, 2y, \underline{\underline{-1}})$$

$PE = f^{-1}(0)$

Cone: $z^2 = x^2 + y^2$

$f(x,y,z) = x^2 + y^2 - z^2$

$df_p = (2x, 2y, -2z)$

$f^{-1}(0)$

$\ni$

$(0,0,0)$

$df_{(0,0,0)} = (0,0,0)$

↳ não sobre

Hiperbolóide de duas folhas: $\boxed{-x^2 - y^2 + z^2 = \boxed{1}}$

$f(x,y,z) = -x^2 - y^2 + z^2$

$df_p = (-2x, -2y, 2z)$

$Hip = f^{-1}(1)$

$p_1 = (0,0,1)$

$p_2 = (0,0,-1)$

$\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ curva $\qquad I = [0,1]$

tq $\alpha(0) = p_1$

$\alpha(1) = p_2$

$\alpha(t) = (x(t), y(t), z(t))$

$z(0) = 1 \qquad z(1) = -1$

Pelo TVI $\exists\, c \in (0,1)$ tq $z(c) = 0$

Daí $\alpha(c) \notin$ Hip

Logo $\nexists$ caminho entre p_1 e p_2 inteiramente contido em Hip.

$\therefore$ Hip é desconexo

(*) $(x-1)^2 + y^2 + z^2 = 1$

$$f(x,y,z) = (x-1)^2 + y^2 + z^2$$

$$df_p = \left(2(x-1),\ 2y,\ 2z\right)$$

$$df_{(0,0,0)} = (-2, 0, 0) \longrightarrow \text{sobrej.}$$

$$df(1,0,0) = (0,0,0) \longrightarrow \nabla!$$

$(1,0,0)$ não satisfaz (*)

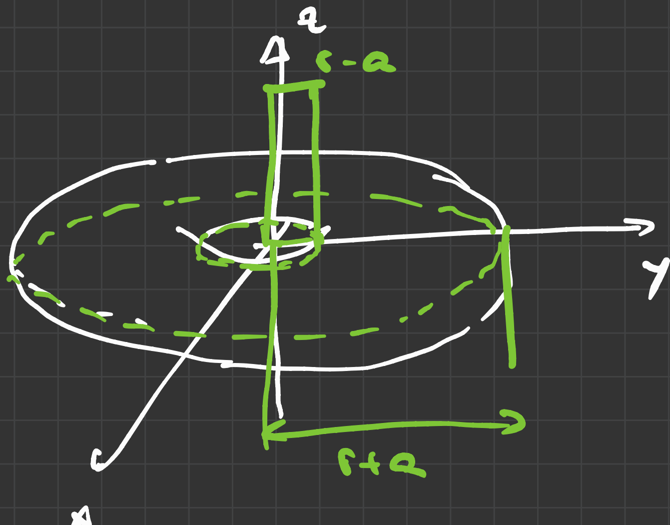
1 é valor reg. de f

$S^2 = f^{-1}(1)$ é sup. reg.

Ex Toro

$$\boxed{z^2 = r^2 - \left(\sqrt{x^2+y^2} - a\right)^2}$$

$a < r$



$z = 0 \rightarrow$ plano xy

$$\left|\sqrt{x^2+y^2} - a\right| = r$$

$$x^2 + y^2 = (r + a)^2$$

$$-\sqrt{x^2+y^2} + a = r$$

$$\sqrt{x^2+y^2} = a - r$$

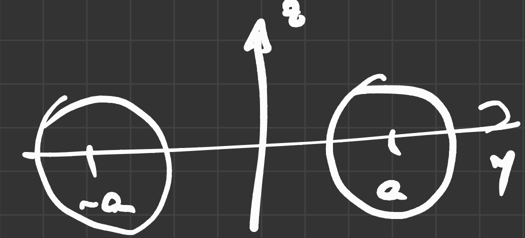
$$x^2 + y^2 = (a - r)^2 = (r - a)^2$$

$\vec{u} = (a, b, 0)$ $\quad ax + by + c = 0$ $\quad c = 0$

$ax + by = 0$

$y = -\frac{a}{b}x$

$n = (1, 0, 0)$

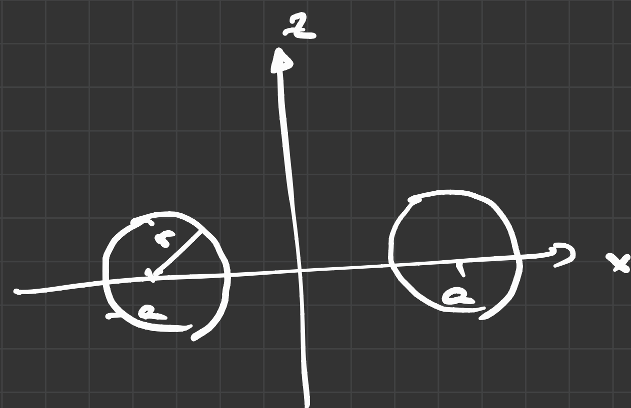


$z^2 = r^2 - \left(\sqrt{x^2 + y^2} - a\right)^2$

$z^2 = r^2 - \left(\pm x \left(\sqrt{1 + \frac{a^2}{b^2}}\right) - a\right)^2$

$\vec{n} = (0, 1, 0)$

$z^2 = r^2 - (\pm x - a)^2$



$(\pm x - a)^2 + z^2 = r^2$

$(x + a)^2 + z^2 = r^2$

$(x - a)^2 + z^2 = r^2$

$$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - a\right)^2 = r^2$$

$\nearrow z \neq 0$

$\searrow z = 0$, $\cancel{\sqrt{x^2+y^2} - a} \neq 0$

$$\frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial f}{\partial x} = 2\left(\sqrt{x^2+y^2} - a\right) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$x^2+y^2 \neq a^2$

$$= \frac{2x\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{2y\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}$$

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

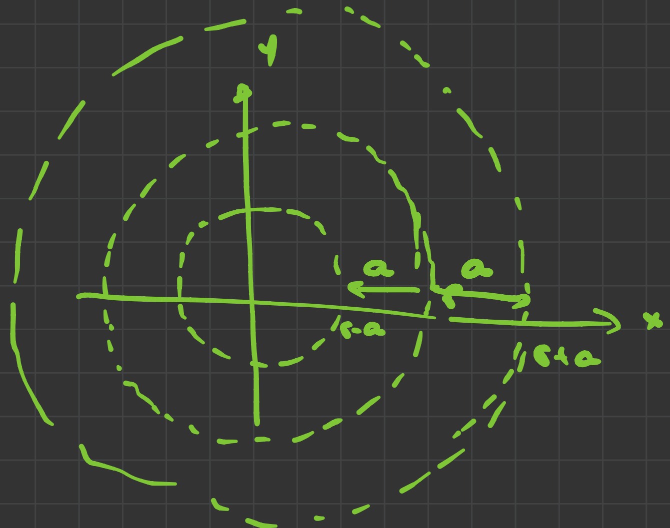
$$\left(\sqrt{x^2+y^2} - a\right)^2 = r^2$$

$$x^2+y^2 = (r \pm a)^2$$

$a = \frac{r}{2}$ $\qquad$ $a = \frac{r}{2}$ $\qquad$ $x^2+y^2 = \frac{r^2}{4}$

$$\frac{\partial f}{\partial x} = \frac{2x\,(r \pm a - a)}{r \pm a}$$

$$\longrightarrow \frac{2x \cdot r}{r+a} \qquad \longrightarrow \frac{2x\,(r-2a)}{r-a}$$



$r = 2$

$a = 1$

$a = \frac{r}{2}$

$r - a = \frac{r}{2}$

$$2^2 + \left(\sqrt{x^2+y^2} - 1\right)^2 = 4$$

Ex 6 (pg 75)

$a \ll r$

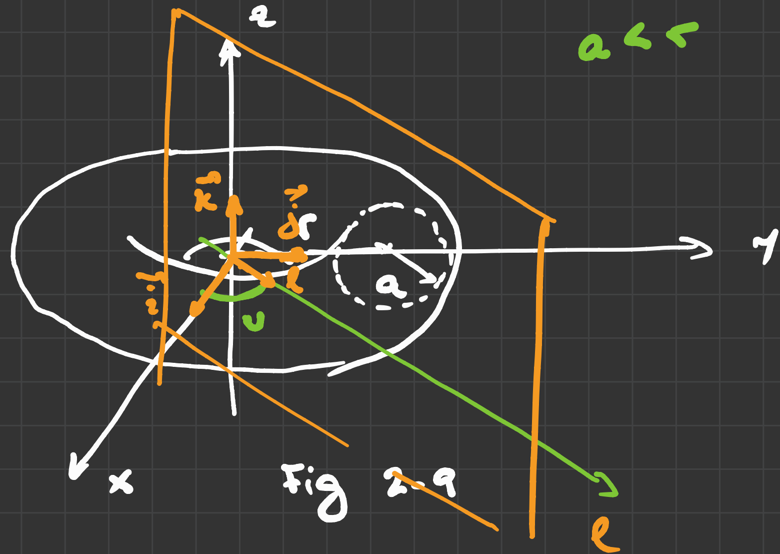
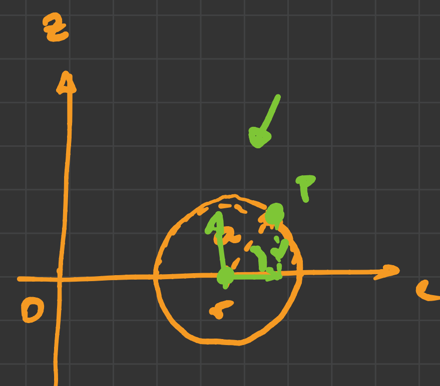


Fig 2.9

$\vec{l} = \cos u \cdot \vec{i} + \operatorname{sen} u \cdot \vec{j}$

$= (\cos u, \operatorname{sen} u, 0)$



$P = r \cdot \vec{l} + a \cos v \, \vec{l} + a \operatorname{sen} v \, \vec{k}$

$P = (r + a\cos v)\,\vec{l} + a \operatorname{sen} v \, \vec{k} = (r + a\cos v)(\cos u, \operatorname{sen} u, 0) + (0, 0, a \operatorname{sen} v)$

$$P = \Big(\underbrace{(r + a\cos v)\cos u}_{x},\ \underbrace{(r + a\cos v)\,\mathrm{sen}\,u}_{y},\ \underbrace{a\,\mathrm{sen}\,v}_{z} \Big)$$

$$u, v \in (0, 2\pi)$$

$$P = \varphi(u,v) \qquad \tilde{\varphi} : \underset{\subset\, \mathbb{R}^2}{(0,2\pi) \times (0,2\pi)} \longrightarrow \underset{T}{\mathbb{R}^3}$$

$$x^2 + y^2 = (r + a\cos v)^2 \cos^2 u + (r + a\cos v)^2\,\mathrm{sen}^2 u \quad (*)$$

$$= (r + a\cos v)^2$$

$$\sqrt{x^2+y^2} = r + a\cos v \longrightarrow a\cos v = \sqrt{x^2+y^2} - r$$

$$a^2 \underbrace{\cos^2 v}_{(1-\mathrm{sen}^2 v)} = \left(\sqrt{x^2+y^2} - r\right)^2 \longrightarrow a^2 - z^2 = \left(\sqrt{x^2+y^2} - r\right)^2$$

$$\boxed{a^2 = z^2 + \left(\sqrt{x^2+y^2} - r\right)^2} \quad (*)$$

$$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - r\right)^2$$

Af: a^2 é valor regular de f $\qquad a < r$

$$T = f^{-1}(a^2)$$

$$\frac{\partial f}{\partial x} = 2\left(\sqrt{x^2+y^2} - r\right) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$$= \frac{2x\left(\sqrt{x^2+y^2} - r\right)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{2y\left(\sqrt{x^2+y^2} - r\right)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial z} = 2z$$

$$z \neq 0 \Rightarrow \frac{\partial f}{\partial z} \neq 0$$

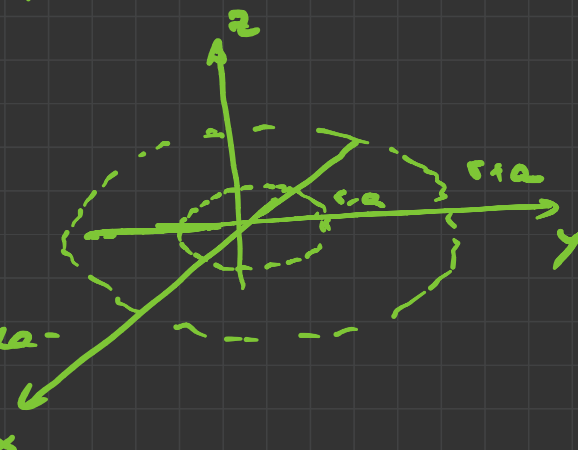
$z = 0$ $\qquad (*) \quad a^2 = z^2 + \left(\sqrt{x^2+y^2} - r\right)^2$

$$a^2 = \left(\sqrt{x^2+y^2} - r\right)^2$$

$$\sqrt{x^2+y^2} - r = \pm a \qquad \sqrt{x^2+y^2} = r \pm a$$

$$x^2+y^2 = (r \pm a)^2$$

$$a < r \Rightarrow r \pm a > 0$$



$$\frac{2x(\pm a)}{r \pm a}$$

$$\parallel$$

x e y não podem ser simultaneamente nulos

Suponha $x \neq 0$

$$\frac{\partial f}{\partial x} = \frac{2x\left(\sqrt{x^2+y^2} - r\right)}{\sqrt{x^2+y^2}} = \frac{2x\left(r \pm a - r\right)}{r \pm a}$$

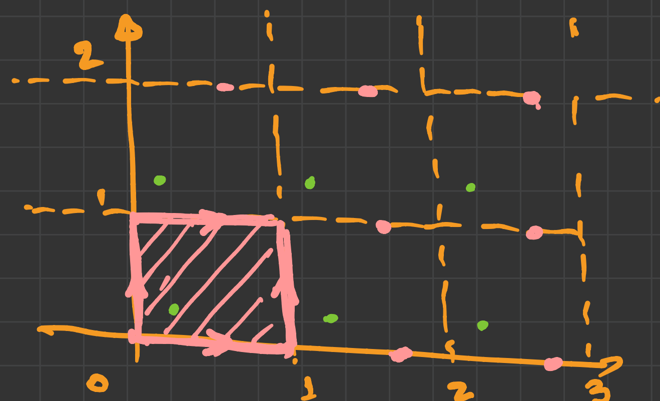
Logo a^2 é valor regular.

Pela prop 2 (aula passada) T é superfície regular.

Obs: $T = \mathbb{R}^2 / \mathbb{Z}^2$

Toro Plano

$(a,b) \sim (c,d) \iff a - c \in \mathbb{Z}$ e $b - d \in \mathbb{Z}$

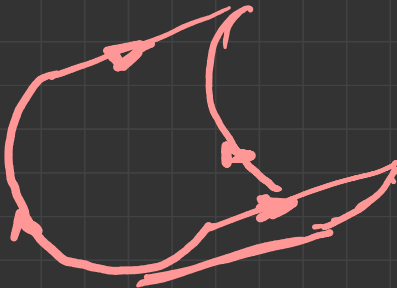


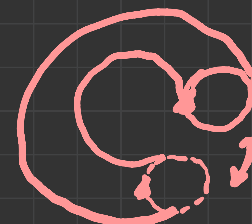
$[a,b] = \{(c,d) \in \mathbb{R}^2 ; (a,b) \sim (c,d)\}$

$\mathbb{R}^2 / \mathbb{Z}^2 = \{[a,b] ; (a,b) \in \mathbb{R}^2\}$

$$\mathbb{R}^2 / \mathbb{Z}^2 \xrightarrow{\tilde{\varphi}} T \subset \mathbb{R}^3$$

↳ difeo





$$\tilde{\varphi} : [0,1]^2 \longrightarrow T$$

$$\tilde{\varphi} = \Big((a\cos(2\pi v) + r)\cos(2\pi u),\ (a\cos(2\pi v) + r)\sin(2\pi u),\ a\sin(2\pi v) \Big)$$

Prop 3: $S \subset \mathbb{R}^3$ sup. reg., $p \in S$.

Então existe viz V de p em S t.q. V é gráfico de uma função de uma das seguintes formas:

$$z = f(x,y) \quad , \quad x = h(y,z) \quad , \quad y = g(x,z)$$

Dem: $\varphi(u,v) = (x(u,v), y(u,v), \boxed{z(u,v)})$

Cond. 3 (Def: Sup. Reg.):

Um dos determinantes Jacobianos:

$$\frac{\partial(x,y)}{\partial(u,v)} \qquad \frac{\partial(y,z)}{\partial(u,v)} \qquad \frac{\partial(x,z)}{\partial(u,v)}$$

é $\neq 0$

Suponha $\boxed{\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0}$

$$\varphi : \underset{\mathbb{R}^2}{\overset{}{U}} \longrightarrow \underset{T}{V}, \quad U \subset \mathbb{R}^2, \quad V \subset T$$

$p \in V$

$\varphi(q) = p$

Seja

$$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x,y)$$

$$(\pi \circ \varphi)^{-1} : V_2 \longrightarrow V_1$$
$$(x,y) \longmapsto (u,v)$$

$$\pi \circ \varphi : U \longrightarrow \mathbb{R}^2$$
$$(u,v) \longmapsto \big(x(u,v), y(u,v)\big)$$

TFI $\Rightarrow \exists$ V_1 viz de q e V_2 aberto de $\mathbb{R}^2$

tq $\pi \circ \varphi \big|_{V_1}$ é inversível e tem inversa diferenciável

vel.

$$f(x,y) = z\left((\pi \circ \varphi)^{-1}(x,y)\right)$$

$\blacksquare$

Prop 4: $p \in S$, S sup. reg.

Seja $\varphi : U \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$, $p \in \varphi(U)$

Suponha q φ satisfaz:

a) Cond 1 (def Sup.)

b) Cond 3 (")

c) φ é bijetora

Então φ^{-1} é contínua (φ satisfaz cond 2).

<u>Dem</u>: $\pi(x,y,z) = (x,y)$

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \text{TFI} \Longrightarrow \underline{\boxed{\pi \circ \varphi}} : V_1 \to V_2 \text{ é invers.}$$

e tem inversa dif.

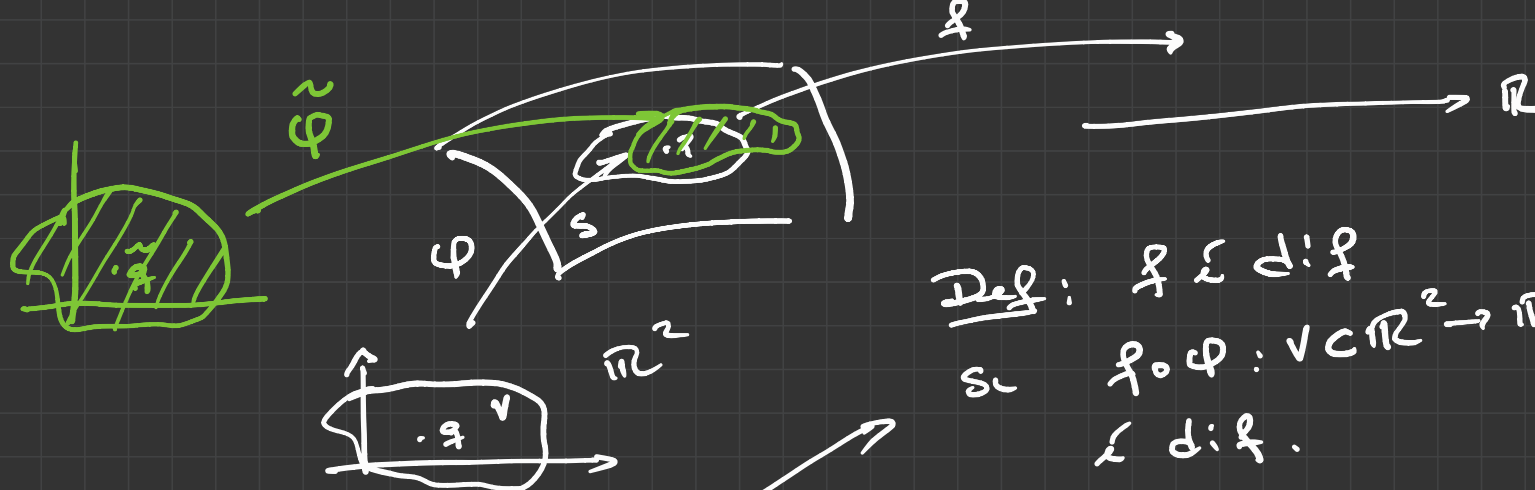
Mas $\varphi^{-1} = \underbrace{(\pi \circ \varphi)^{-1}} \circ \underset{\sim}{\pi}$. Logo φ^{-1} é cont. $\boxtimes$

Diferenciabilidade em S

$$f: \mathbb{R}^n \longrightarrow S \subset \mathbb{R}^3$$

↳ Sabemos o q. é ser difer.

$$f: S \supset U \longrightarrow \mathbb{R}$$



Def: f é dif se $f \circ \varphi : V \subset \mathbb{R}^2 \to \mathbb{R}$ é dif.

$$f \circ \tilde{\varphi} = (f \circ \varphi) \circ \underbrace{(\varphi^{-1} \circ \tilde{\varphi})}_{\text{dif}}$$

<u>Prop 1 (2.3 Manfredo)</u>

<u>[Mudança de parâmetros]</u> : $p \in S$, S sup. reg.

Sejam $\varphi_1 : U \subset \mathbb{R}^2 \longrightarrow S$ e $\varphi_2 : V \subset \mathbb{R}^2 \longrightarrow S$

duas parametrização tq $p \in \underbrace{\varphi_1(U) \cap \varphi_2(V)}_{W} \subset S$

Então $h = \varphi_1 \circ \varphi_2^{-1} : \varphi_2^{-1}(W) \longrightarrow \varphi_1(W)$

é difeomorfismo (é dif e tem inversa dif.).