

THEOREM 6.7. Given m parallel to l such that m does not contain a limiting parallel ray to l in either direction. Then there exists a common perpendicular to m and l (which is unique by Theorem 6.5).

This theorem is proved by Borsuk and Szmielew (1960, p. 291) by a continuity argument, but their proof gives you no idea of how to actually find the common perpendicular. There is an easy way to find it in the Klein and Poincaré models, discussed in the next chapter. Hilbert gave a direct construction, which we will sketch. (Project 1 gives another.)

Proof:

Hilbert's idea is to find two points H and K on l that are equidistant from m , for once these are found, the perpendicular bisector of segment HK is also perpendicular to m (see Lemma 6.2). Choose any two points A and B on l and suppose that the perpendicular segment AA' from A to m is longer than the perpendicular segment BB' from B to m . (See Figure 6.16.) Let E be the point between A' and A such that $A'E \cong B'B$. On the same side of AA' as B , let \overrightarrow{EF} be the unique ray such that $\sphericalangle A'EF \cong \sphericalangle B'BG$, where $A * B * G$. The key point that will be proved in Major Exercises 2–6 is that \overrightarrow{EF} intersects \overrightarrow{AG} in a point H . Let K be the unique point on \overrightarrow{BG} such that $EH \cong BK$. Drop perpendiculars $\overrightarrow{HH'}$ and $\overrightarrow{KK'}$ to m . The upshot of these constructions is that $\square EHH'A'$ is congruent to $\square BKK'B'$ (just divide them into triangles). Hence, the corresponding sides HH' and KK' are congruent, so that the points H and K on l are equidistant from m , as required. ■

To sum up, given a point P not on l , there exist exactly two limiting parallel rays to l through P , one in each direction. There are infinitely many lines through P that do not enter the region between the limiting

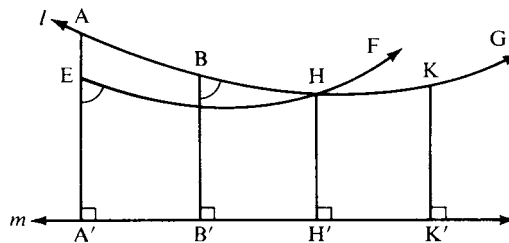


FIGURE 6.16

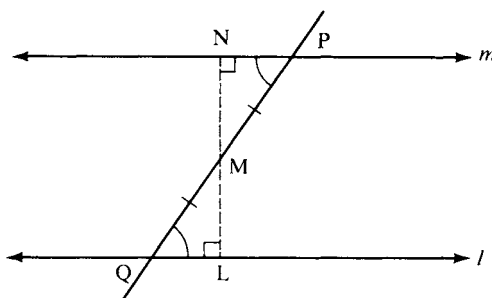


FIGURE 6.27

- perpendicular to l and \overleftrightarrow{AC} ; see Major Exercise 7 for the asymptotically parallel case.)
14. In Theorem 4.1 it was proved in neutral geometry that if alternate interior angles are congruent, then the lines are parallel. Strengthen this result in hyperbolic geometry by proving that the lines are divergently parallel, i.e., that they have a common perpendicular. (Hint: Let M be the midpoint of transversal segment PQ and drop perpendiculars MN and ML to lines m and l ; see Figure 6.27. Prove that L , M , and N are collinear by the method of congruent triangles.)
 15. *Make a long list of statements equivalent in neutral geometry to Hilbert's parallel postulate.* This list is a reward for all the work you have done.
 16. Although the circumscribed circle may not exist for some triangles in hyperbolic geometry, prove that the inscribed circle always exists. (Hint: Verify that the usual Euclidean proof—that the angle bisectors meet in a point equidistant from the sides—still works. Use the crossbar theorem.)
 17. Comment on the following injunction by Saint Augustine: “The good Christian should beware of mathematicians and all those who make empty prophesies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell.”

MAJOR EXERCISES

1. Let A , D be points on the same side of line \overleftrightarrow{BC} such that $\overleftrightarrow{BA} \parallel \overleftrightarrow{CD}$. Then the figure consisting of segment BC (called the *base*) and rays \overrightarrow{BA} and \overrightarrow{CD} (called the *sides*) is called the *biangle* $[ABCD]$ with *vertices* B and C

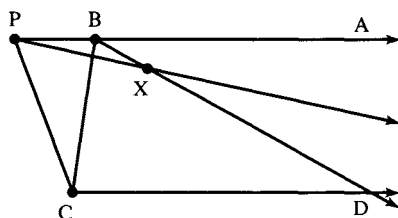


FIGURE 6.28

(see Figure 6.28). The interior of $[ABCD$ is the intersection of the interiors of its angles $\sphericalangle ABC$ and $\sphericalangle DCB$; if P lies in the interior and X is either vertex, ray \overrightarrow{XP} is called an interior ray. We write $\overrightarrow{BA} \parallel \overrightarrow{CD}$ when these rays are sides of a biangle and when every interior ray emanating from B intersects \overrightarrow{CD} ; in that case, we say that \overrightarrow{BA} is limiting parallel to \overrightarrow{CD} , generalizing the previous definition which required $\sphericalangle DCB$ to be a right angle, and we say that the biangle $[ABCD$ is closed at B . Given $\overrightarrow{BA} \parallel \overrightarrow{CD}$, prove the following generalization of Exercise 7: If $P * B * A$ or if $B * P * A$, then $\overrightarrow{PA} \parallel \overrightarrow{CD}$.

2. Symmetry of limiting parallelism. If $\overrightarrow{BA} \parallel \overrightarrow{CD}$, then $\overrightarrow{CD} \parallel \overrightarrow{BA}$. (In that case we say simply that biangle $[ABCD$ is closed.) Justify the unjustified steps in the proof (see Figure 6.29).

Proof:

(1) Assume that $[ABCD$ is not closed at C . (2) Then some interior ray \overrightarrow{CE} does not intersect \overrightarrow{BA} . (3) Point E , which so far is just a label, can be chosen so that $\sphericalangle BEC < \sphericalangle ECD$, by the important corollary to Aristotle's axiom, Chapter 3. (4) Segment BE does not intersect \overrightarrow{CD} . (5) Interior ray \overrightarrow{BE} intersects \overrightarrow{CD} in a point F , and $B * E * F$. (6) Since $\sphericalangle BEC$ is an exterior angle for $\triangle EFC$, $\sphericalangle BEC > \sphericalangle ECF$. (7) Contradiction. (I am indebted to George E. Martin for this simple proof.) ■

3. Transitivity of limiting parallelism. If \overrightarrow{AB} and \overrightarrow{CD} are both limiting parallel to \overrightarrow{EF} , then they are limiting parallel to each other. Justify the steps in the proof. (See Figure 6.30.)

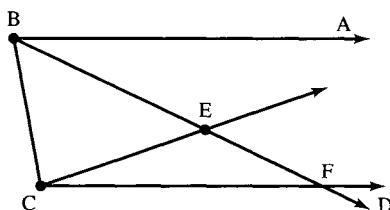


FIGURE 6.29

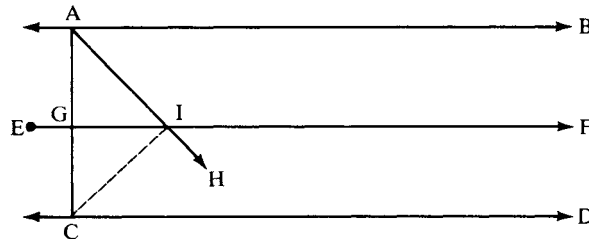


FIGURE 6.30

Proof:

(1) \overleftrightarrow{AB} and \overleftrightarrow{CD} have no point in common. (2) Hence, there are two cases, depending on whether \overleftrightarrow{EF} is between \overleftrightarrow{AB} and \overleftrightarrow{CD} or \overleftrightarrow{AB} and \overleftrightarrow{CD} are both on the same side of \overleftrightarrow{EF} . (3) In case \overleftrightarrow{EF} is between \overleftrightarrow{AB} and \overleftrightarrow{CD} , let G be the intersection of \overleftrightarrow{AC} with \overleftrightarrow{EF} . We may assume G lies on ray \overrightarrow{EF} ; otherwise we can consider \overrightarrow{GF} . (4) Any ray \overrightarrow{AH} interior to $\sphericalangle GAB$ must intersect \overleftrightarrow{EF} in a point I . (5) \overrightarrow{IH} , lying interior to $\sphericalangle CIF$, must intersect \overleftrightarrow{CD} . (6) Hence, any ray \overrightarrow{AH} interior to $\sphericalangle CAB$ must intersect \overleftrightarrow{CD} , so \overleftrightarrow{AB} is limiting parallel to \overleftrightarrow{CD} . ■

Step (7) is the following sublemma. That this requires such a long proof was overlooked even by Gauss. The proof (for which I am indebted to Edwin E. Moise) uses our hypotheses of limiting parallelism. If we had made the weaker hypothesis of just parallel lines, the sublemma would not follow, as you will show in Exercise K-2(c) of Chapter 7.

SUBLEMMA. If \overleftrightarrow{AB} and \overleftrightarrow{CD} are both on the same side of \overleftrightarrow{EF} , we may assume that \overleftrightarrow{CD} , for example, is between \overleftrightarrow{AB} and \overleftrightarrow{EF} (see Figure 6.31).

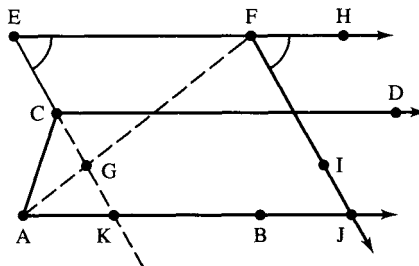


FIGURE 6.31

Proof of Sublemma:

(1) It suffices to prove there is a line transversal to the three rays \vec{AB} , \vec{CD} , \vec{EF} . (2) In case A and F are on the same side of \vec{EC} , then ray \vec{EA} is interior to $\sphericalangle E$. (3) Then \vec{EA} intersects \vec{CD} , by symmetry. (4) So \vec{EA} is our transversal. (5) In case A and F are on opposite sides of \vec{EC} , let G be the point at which AF meets \vec{EC} . (6) Choosing H such that $E * F * H$, we have $\vec{FH} \parallel \vec{AB}$. (7) $\sphericalangle HFG > \sphericalangle E$. (8) Therefore there is a ray \vec{FI} interior to $\sphericalangle HFA = \sphericalangle HFG$ such that $\sphericalangle HFA \cong \sphericalangle E$. (9) \vec{FI} meets \vec{AB} at a point J. (10) $\vec{FJ} \parallel \vec{EC}$. (11) \vec{EC} intersects side AF and does not intersect side FJ of $\triangle AFJ$. (12) Hence \vec{EC} intersects AJ and is our transversal. ■

Conclusion of Proof (see Figure 6.32):

(8) Then AE intersects \vec{CD} in a point G, which we may assume lies on ray \vec{CD} . (9) Any ray \vec{AH} interior to $\sphericalangle GAB$ intersects \vec{EF} in a point I. (10) Since \vec{CD} enters $\triangle AEI$ at G and does not intersect side EI, it must intersect AI. (11) Therefore, \vec{CD} is limiting parallel to \vec{AB} . ■

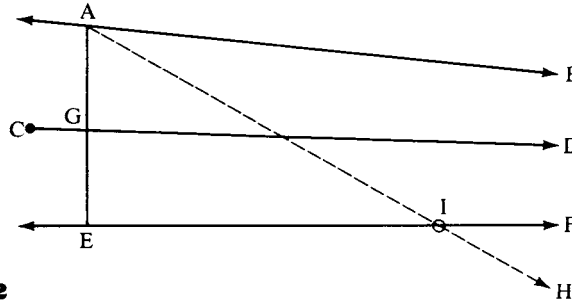


FIGURE 6.32

Note 1. The last four steps did not use the hypothesis that $\vec{CD} \parallel \vec{EF}$; they therefore prove that *any line between two asymptotically parallel lines is asymptotically parallel to both and in the same direction.*

Note 2. Given rays r and s , define $r \sim s$ to mean that either $r \subset s$ or $s \subset r$ or $r \parallel s$. Major Exercises 1–3 show that this is an equivalence relation among rays. An equivalence class of rays is called an *ideal point*, or an *end*, and we adopt the convention that it lies on all (and only those) lines containing the rays making up the class. Since a point on a line breaks the line into two opposite rays and opposite rays are not equiva-

lent, we see that every line has two ends lying on it. The set of all ideal points was named by Cayley *the absolute*. (This is the beginning of constructing a hyperbolic analogue of the projective completion of an affine plane described in Chapter 2; we continue the construction in Major Exercise 13. The absolute is analogous to the line at infinity of the affine plane, but the absolute could not be a new line, because it intersects each old line in two points; it will turn out to be a conic in the projective completion.)

If R, S are the vertices of r, s , where $r \parallel s$, and Ω is the ideal point determined by these rays, we write $r = P\Omega$ and $s = S\Omega$ and refer to the closed biangle with sides r, s as the *singly asymptotic triangle* $\Delta RS\Omega$. The next two exercises show that these triangles have some properties in common with ordinary triangles. (You can similarly define as an exercise *doubly* (two ideal points) and *triply* (three ideal points) *asymptotic triangles*.)

4. *Exterior angle theorem.* If $\Delta PQ\Omega$ is a singly asymptotic triangle, the exterior angles at P and Q are greater than their respective opposite interior angles. Justify the steps in the proof.

Proof (see Figure 6.33):

(1) Given $\vec{R} * Q * P$. We must show that $\sphericalangle RQ\Omega$ is greater than $\sphericalangle QP\Omega$.
 (2) Let \vec{QD} be the unique ray on the same side of \vec{PQ} as ray $\vec{Q}\Omega$ such that $\sphericalangle RQD \cong \sphericalangle QP\Omega$. (3) If $U * Q * D$, then $\sphericalangle UQP \cong \sphericalangle QP\Omega$. (4) By Exercise 14, \vec{QD} is divergently parallel to $\vec{P}\Omega$. (5) Hence, \vec{QD} is between \vec{QR} and $\vec{Q}\Omega$. (6) $\sphericalangle RQ\Omega > \sphericalangle QP\Omega$. ■

5. *Congruence theorem.* If in asymptotic triangles $\Delta AB\Omega$ and $\Delta A'B'\Omega'$ we have $\sphericalangle BA\Omega \cong \sphericalangle B'A'\Omega'$, then $\sphericalangle AB\Omega \cong \sphericalangle A'B'\Omega'$ if and only if

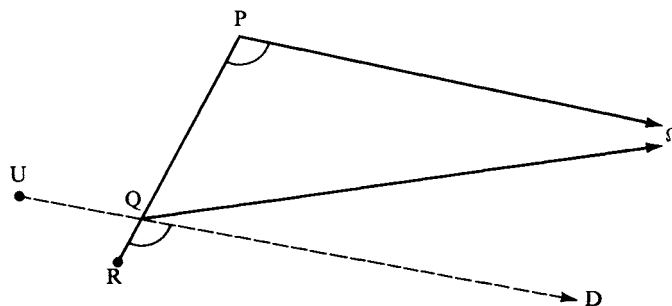


FIGURE 6.33

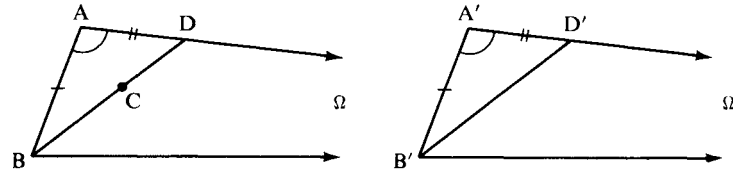


FIGURE 6.34

$AB \cong A'B'$. Justify the steps in the proof and deduce as a corollary that $PQ \cong P'Q'$ if and only if $\Pi(PQ)^\circ = \Pi(P'Q')^\circ$.

Proof (see Figure 6.34):

- (1) Assume $AB \cong A'B'$ and on the contrary $\sphericalangle AB\Omega > \sphericalangle A'B'\Omega'$. (2) There is a unique ray \vec{BC} between $B\Omega$ and \vec{BA} such that $\sphericalangle ABC \cong \sphericalangle A'B'\Omega'$. (3) \vec{BC} intersects $A\Omega$ in a point D . (4) Let D' be the unique point on $A'\Omega'$ such that $AD \cong A'D'$. (5) Then $\triangle BAD \cong \triangle B'A'D'$. (6) Hence, $\sphericalangle A'B'D' \cong \sphericalangle A'B'\Omega'$, which is absurd. (7) Assume conversely that $\sphericalangle AB\Omega \cong \sphericalangle A'B'\Omega'$ and on the contrary $A'B' < AB$. (8) Let C be the point on AB such that $BC \cong B'A'$, and let $C\Omega$ be the ray from C limiting parallel to $A\Omega$ (see Figure 6.35). (9) Then $C\Omega$ is also limiting parallel to $B\Omega$. (10) By the first part of the proof, $\sphericalangle BC\Omega \cong \sphericalangle B'A'\Omega'$; hence, $\sphericalangle BC\Omega \cong \sphericalangle BA\Omega$. (11) But $\sphericalangle BC\Omega > \sphericalangle BA\Omega$, which is a contradiction. ■

6. *Conclusion of the proof of theorem 6.7.* We wish to show that \vec{EF} intersects \vec{AG} (see Figure 6.36). Justify the steps in the proof.

Proof:

- (1) Let $\vec{A'M}$ be limiting parallel to \vec{EF} , $\vec{A'N}$ limiting parallel to \vec{AG} , and $\vec{B'P}$ limiting parallel to \vec{BG} . (2) Since $EA' \cong BB'$ and $\sphericalangle A'EF \cong \sphericalangle B'BG$,

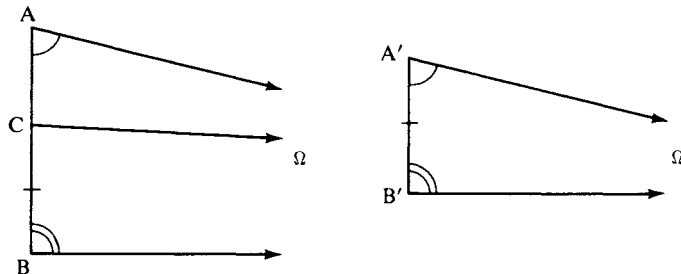


FIGURE 6.35

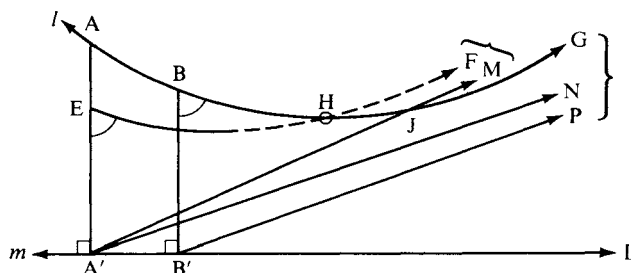


FIGURE 6.36

we have $\sphericalangle EA'M \cong \sphericalangle BB'P$. (3) $\vec{B'L}$ differs from $\vec{B'P}$ and $\vec{A'L}$ differs from $\vec{A'N}$. (4) $\sphericalangle MA'L \cong \sphericalangle PB'L$. (5) $\vec{B'P}$ is limiting parallel to $\vec{A'N}$. (6) Hence, $\sphericalangle NA'L$ is smaller than $\sphericalangle PB'L$. (7) It follows that $\vec{A'M}$ lies between $\vec{A'N}$ and $\vec{A'A}$, so it must intersect \vec{AG} in a point J. (8) J is on the same side of \vec{EF} as A' ; hence, it is on the side opposite from A. (9) Thus, AJ intersects \vec{EF} in a point H, which must be on \vec{EF} because H is on the same side of $\vec{AA'}$ as J. ■

Where was the hypothesis of this theorem used?

7. In Exercises 12 and 13 we considered the perpendicular bisectors of the sides of $\triangle ABC$ and we showed that (1) if two of them have a common point, the third passes through that point; (2) if two of them have a common perpendicular, the third has that same perpendicular. It follows that if two of them are asymptotically parallel, then any two of them are asymptotically parallel. This result can be strengthened as follows: if perpendicular bisectors l and m are asymptotically parallel in the direction of ideal point Ω , then the third perpendicular bisector n is asymptotically parallel to l and m in the same direction Ω . Give the proof and justify each step. The proof is based on the following two lemmas:

LEMMA 6.3. Given $\triangle ABC$. Let l , m , and n be the perpendicular bisectors of sides AB , BC , and AC at their midpoints L , M , and N , respectively. Let $AC \cong AB$ and $AC \cong BC$ (AC is the longest side). Then l , m , and n all intersect AC .

Proof:

(1) $\sphericalangle B \cong \sphericalangle A$ and $\sphericalangle B \cong \sphericalangle C$. (2) Hence, there is a point L' on AC such that $\sphericalangle A \cong \sphericalangle L'BA$, and a point M' on AC such that $\sphericalangle C \cong \sphericalangle M'BC$. (See Figure 6.37.) (3) Then $AL' \cong BL'$ and $CM' \cong BM'$. (4)