where $h(a)$ is defined in Exercise 1. We see that $\phi(a, b, \gamma)$ is Lipschitz continuous in γ for fixed a and b. If two triangles have the same value of γ and the same value of a or b, with the other varying, we have, again by the triangle inequality, that

$$|\phi(a_2, b, \gamma) - \phi(a_1, b, \gamma)| \leq |a_2 - a_1|,$$
$$|\phi(a, b_2, \gamma) - \phi(a, b_1, \gamma)| \leq |b_2 - b_1|.$$

We see that $\phi(a, b, \gamma)$ is Lipschitz continuous in each of its variables. For the domain of this function, we have assumed that a, b, γ are in open intervals $(0, \infty), (0, \pi)$. The investigation of limiting cases is left to the reader, as is the investigation of the other criterion functions. The conclusion is the following:

Theorem 2.10: In triangle congruences, the remaining sides and angles depend continuously on the given sides and angles.

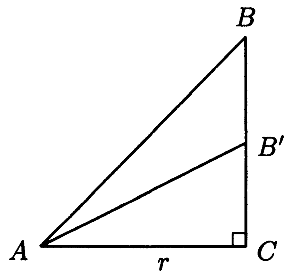

Fig. 2.5e.

Exercises

1. First consider thin *right* triangles by starting with triangle ABC as in Fig. 2.5e, with $|AC| = |BC| = r$, $\angle C = 90°$, and $\angle A = \angle B = \alpha$; α depends only on r. Bisect the angle $\angle A$, letting B' be the point between B and C where the bisector meets the opposite side. Then, by a suitable construction, by use of one of the congruence criteria, and by Theorem 2.2, show that $|B'C|$ is less than half of $|BC| = r$. Then bisect the angle at A of the triangle $AB'C$, and so on, thus producing right triangles with angle at A equal to $2^{-n}\alpha$, $n = 1, 2, \ldots$. Denote the length of the side opposite to A by $\tau(r, 2^{-n}\alpha)$, so that

$$\tau(r, 2^{-n}\alpha) < 2^{-n}r = (r/\alpha)2^{-n}\alpha.$$

For any angle γ less than α choose n so that $2^{-n-1}\alpha < \gamma \leq 2^{-n}\alpha$, and show that $\tau(r,\gamma) < 2r/\alpha\ 2^{-n-1}\alpha < 2r/\alpha\ \gamma = h(r)\gamma$, where $h(r)$ stands for $2r/\alpha$.

2. By a further construction and further use of Theorem 2.2, show that the third side of an isosceles triangle with two sides of length r and included angle γ is less than $\tau(r,\gamma)$, and conclude that $\phi(r,r,\gamma) < h(r)\gamma$, as claimed in Lemma 2.2. *Hint:* Use one of the congruence criteria to show that the bisector of the angle γ is perpendicular to the opposite side.
3. Show the Lipschitz continuity of the right triangle functions defined in terms of Fig. 2.5f as

$$\sigma(c,\alpha) = a, \qquad \kappa(c,\alpha) = b.$$

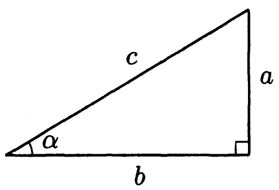

Fig. 2.5f.

4. Show that the functions σ and κ are meaningful for the limiting values $\alpha = 0°$ and $\alpha = 90°$, and give their limiting values for those angles.

2.6 Intersections of Lines and Circles

The distance from a point to a line (if not zero) is defined as the length of the perpendicular ZB of Theorem 2.2. If the point is on the line, the distance is of course taken as zero. The *circle of radius r (> 0) centered at P* is defined (as expected) to be the set of points Q such that $|PQ| = r$. A point Q is *inside* the circle if $|PQ| < r$ and *outside* the circle if $|PQ| > r$. A line is *tangent* to a circle if it has exactly one point in common with the circle, that is, if it has at least one point in common with the circle and is perpendicular to the line through that common point and the center of the circle. There is a tangent at every point of a circle. Two distinct circles are *tangent* if they intersect at a single point. Then, according to Exercise 2 below, they have a common tangent line at that point.

Theorem 2.11: If the distance from the center of a circle to a line is less than the radius of the circle, the line intersects the circle in just two points. If the